JOURNAL OF COMPUTATIONAL PHYSICS 10, 22–39 (1972)

An Integro-Differential Equation Approach to Acoustic Scattering from Fluid-Immersed Elastic Bodies

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An integro-differential equation approach for solving problems of acoustic scattering and radiation from fluid-immersed elastic bodies is described. A consistent set of integral and differential equations relating the incident pressure field to the surface pressure and displacements is developed for a submerged elastic spherical shell and results obtained using a numerical solution technique are compared with the classical modal expansion solution.

I. INTRODUCTION

The evaluation of acoustic wave scattering from complex shaped objects has become realizable in the past few years due to the increasing availability of digital computers. In particular, integral equation techniques have received increasing attention because of the flexibility which they bring to this general problem area.

While the numerical solution of the acoustic problem for rigid and free-surface bodies is relatively straightforward, the corresponding problem of acoustic scattering from elastic shells is by comparison much more difficult and has received less attention. Certainly a great deal of success has been achieved in the analysis of elastic structures from the viewpoint of vibration analysis, but the related problem of the fluid-coupled elastic structure has apparently not been as extensively treated.

Our concern in this presentation will be the outline of an integro-differential equation (IDE) approach to the scattering of a time-harmonic plane acoustic wave from an elastic spherical shell. Numerical results obtained from the IDE will be validated by comparison with the classical series solution obtained from the separation of variables approach. An advantage of the IDE is its extendability to more general geometrical shapes. Thus, while the numerical results which follow are restricted to the spherical case so that the validity of the IDE approach can be established by comparison with the rigorously correct solution, their real signifi-

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cance lies in the inherent potential of the method for the treatment of more general, and thus more practically interesting, body shapes. Integro-differential equations have, it should be noted, been numerically treated elsewhere in different contexts (Kogan, 1969; Khabbaze, 1970).

II. Mathematical Development

The analytic solution, i.e., a closed form expression in terms of standard functions (which may involve series with infinite indices), of the three dimensional elastic shell problem can be carried out for only the limited number of shapes whose surfaces coincide with constant coordinate surfaces in separable coordinate systems (Morse and Feshbach, 1953). Consequently, the analysis of more general three-dimensional elastic shells requires the availability of a method which is not restricted to certain geometries.

The integral equation approach which has found widespread use in electromagnetic theory (Poggio and Miller, 1970), is also well suited to the acoustics regime (see Schenck, 1968, for example). It is an integral equation method to which this paper is primarily devoted. We will, however, make use of an analytical approach in order to make available independent data for the validation of results.

A. *The Integral Representation for Acoustic Scattering*

Our main interest in this section is to derive an integral equation relating the acoustic fields over a surface to some driving source. Let us consider then the

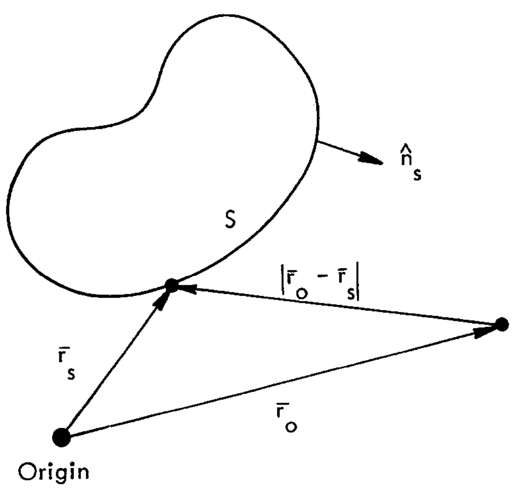

FIG. 1. Geometry for integral representations.

geometry shown in Fig. 1. The volume V is a source-free region containing a homogeneous fluid of density ρ exterior to the surface S. It is well known that the pressure $p(\bar{r}_0)$ at a point $\bar{r}_0$ within V but not on S is given by (Baker and Copson, 1950)

$$p(\bar{r}_0) = \frac{1}{4\pi} \int_S \left\{ p(\bar{r}_s) \frac{\partial}{\partial n_s} G(\bar{r}_0, \bar{r}_s) + i\omega\rho\dot{w}(\bar{r}_s) G(\bar{r}_0, \bar{r}_s) \right\} dA_s(\bar{r}_s), \qquad (1)$$

with $\bar{r}_s$ representing a point on the surface S, dA_s the differential element of area, $\hat{n}_s$ a unit normal from the surface into the volume, $\dot{w}$ the normal velocity, and G, the Green's function given by

$$G(\bar{r}_0, \bar{r}_s) = \exp[-ik \mid \bar{r}_0 - \bar{r}_s \mid]/\mid \bar{r}_0 - \bar{r}_s \mid .$$

k is the acoustic wave number, i.e., $k = \omega/c$ where ω is the angular frequency ($e^{i\omega t}$ time variation assumed), and c the propagation velocity in the fluid. The implication of Eq. (1) is that the pressure at any point in space outside S can be found by a surface integration of the pressure and velocity distributions over S. Hence, $p(\bar{r}_s)$ and $\dot{w}(\bar{r}_s)$ are oftentimes referred to as source distributions on S, over which they satisfy the boundary condition

$$\partial p(\bar{r}_s)/\partial n_s = -i\omega\rho w(\bar{r}_s). \qquad (2)$$

If the surface S represents a passive scatterer then the surface source distributions must be induced by some external mechanism. In the present analysis we consider that there exists a pressure field denoted by $p_{\text{inc}}(\bar{r}_0)$, i.e., an incident pressure field, which induces the equivalent sources on S. Then the total pressure at any point outside S is given by

$$p(\bar{r}_0) = p_{\text{inc}}(\bar{r}_0) + \frac{1}{4\pi} \int_S \left\{ p(\bar{r}_s) \frac{\partial}{\partial n_s} G(\bar{r}_0, \bar{r}_s) + i\omega\rho\dot{w}(\bar{r}_s) G(\bar{r}_0, \bar{r}_s) \right\} dA_s(\bar{r}_s) \quad (3)$$

with $\bar{r}_0$ within V but not on S.

In general, the pressure and velocity distributions due to an impinging wave on a surface are unknown and so must be found before the total pressure field in the fluid can be determined. One method of accomplishing this is the derivation of an integral equation from (3) by allowing the observation point $\bar{r}_0$ to approach the surface S. The contribution of the integral term in (3) in the limit as $\bar{r}_0 \to S$ can be expressed as [Kellog, 1953, p. 167]

$$\frac{p(\bar{r}_0)}{2} + \frac{1}{4\pi} ⨍_S \left\{ p(\bar{r}_s) \frac{\partial}{\partial n_s} G(\bar{r}_0, \bar{r}_s) + i\omega\rho\dot{w}(\bar{r}_s) G(\bar{r}_0, \bar{r}_s) \right\} dA_s(\bar{r}_s),$$

where $⨍_S$ is the principal value integral defined as $⨍_S (\ ) \, d\sigma = \lim_{\Delta S \to 0} \int_{S-\Delta S} (\ ) \, d\sigma$ in which ΔS is a small element of area about the point $\bar{r}_0 = \bar{r}_s$.

In view of this limiting procedure, Eq. (3) becomes

$$p(\bar{r}_0) = 2p_{\text{inc}}(\bar{r}_0) + \frac{1}{2\pi} \int_S \left\{ p(\bar{r}_s) \frac{\partial}{\partial n_s} G(\bar{r}_0, \bar{r}_s) + i\omega\rho\dot{w}(\bar{r}_s)\, G(\bar{r}_0, \bar{r}_s) \right\} dA_s(\bar{r}_s), \quad (4)$$

with $\bar{r}_0 \in S$, i.e., $\bar{r}_0$ on the surface S. Equation (4) is the desired integral equation for the unknown surface pressure p in terms of the incident pressure field and the normal velocity over S. It remains to relate p and $\dot{w}$ as functions of the surface coordinates to allow a solution of (4) for either p or $\dot{w}$. (While Eq. (2) could evidently be used to reduce Eq. (4) to an integral equation involving p and $\partial p/\partial n_s$ alone, the mixed boundary value problem this represents would still require a treatment equivalent to that which follows to find the influence of the shell boundary on the surface distribution of the induced sources. We thus choose to continue with a formulation in terms of p and $\dot{w}$).

B. *The Integro-Differential Equation Approach*

Because of our interest in scattering by a spherical shell of an axially incident plane wave, let us specialize Eq. (4) to this geometry (see Fig. 2).

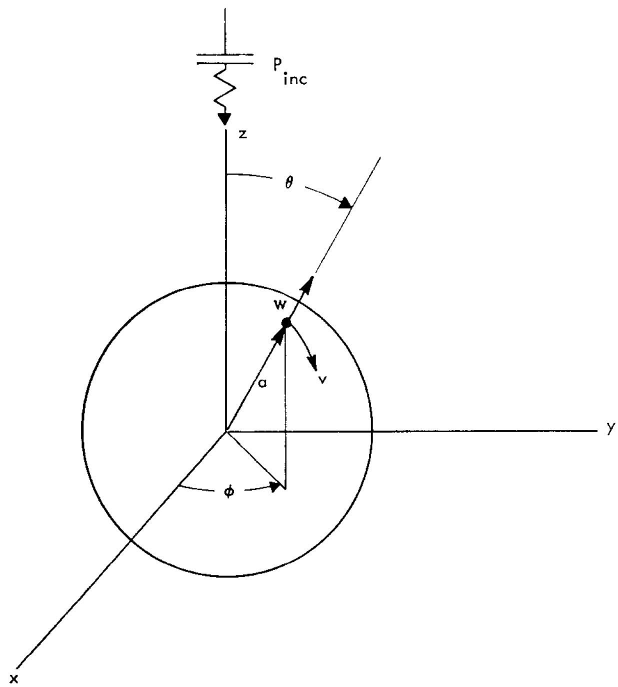

FIG. 2. Geometry for scattering by a spherical shell.

Thus, for a sphere of radius a we obtain

$$-p_{\text{inc}}(a, \theta) = -\frac{1}{2} p(a, \theta) + \frac{1}{4\pi} \int \left\{ p(a, \theta') \frac{\partial}{\partial n_s'} G - \omega^2 \rho w(a, \theta') G \right\} a^2 \, d\varphi' \, d\theta' \sin \theta', \quad (5)$$

where the azimuthal symmetry in φ about the polar axis is exploited by writing p and w as functions of θ only. Since no simple local relationship exists between p and w, which globally interact over S, an impedance boundary condition approximation cannot be used to reduce (5) to an equation in p (or w) alone. An additional equation which relates p and w is necessary to complete the formulation of the scattering process for an elastic spherical shell.

Let us restrict in this analysis our attention to a spherical shell of thickness Δ where the shell is thin enough such that: (1) Radial normal stresses and bending moments can be neglected; and (2) The deformations fall within the regime of the linear equations of motion. Baker (1961) and Baker *et al.* (1966) present the equations of motion for such a spherical shell of radius a and density ρ_s. Using the geometrical definitions of Fig. 2 (with v the θ-directed surface displacement and w the radially directed surface displacement) these can be written as

$$\frac{d^2 v}{d\theta^2} + \cot \theta \frac{dv}{d\theta} + (A\omega^2 - \nu - \cot^2 \theta) v + (1 + \nu) \frac{dw}{d\theta} = 0, \quad (6)$$

$$\frac{dv}{d\theta} + v \cot \theta + \left\{ 2 - \frac{A\omega^2}{1 + \nu} \right\} w + Bp = 0, \quad (7)$$

where ν is Poisson's ratio, $B = (1 - \nu) a^2/Y\Delta$, Y is Young's modulus, and $A = [(1 - \nu^2)/Y] \rho_s a^2$.

The two differential Eqs. (6) and (7), together with integral Eq. (5), provide three equations for the unknown quantities v, w and p. Taken together, these equations form a coupled set for the spherical elastic shell. An attractive alternative is available by solving (6) and (7) for v in terms of $dw/d\theta$ and $dp/d\theta$ and by substituting the resulting expression into (7). One then arrives at a single differential equation of the form

$$E(d^2 w/d\theta^2) + E \cot \theta (dw/d\theta) + Dw + F(d^2 p/d\theta^2) + F \cot \theta (dp/d\theta) + Bp = 0, \quad (8)$$

where

$$E = [\nu - 1 + A\omega^2/(1 + \nu)]/(\nu - 1 - A\omega^2),$$
$$F = B/(\nu - 1 - A\omega^2),$$
$$D = 2 - [A\omega^2/(1 + \nu)].$$

Either of the sets [Eqs. (5)–(7) or Eqs. (5) and (8)] shall be referred to as the integro differential equation for the scattering problem. It should be pointed out here that although Equation (4) is a general equation which pertains to any geometry, the equations of motion are more restricted in their applicability to the spherical shell case only. For complete generality it would be necessary to formulate the differential equations of motion for an arbitrarily shaped shell.

The pressure field scattered by the obstacle S can then be evaluated by substituting the surface pressure and displacement into Eq. (1). The far-zone radiation field f_∞ is obtained from (1) in the usual way, i.e.,

$$f_\infty = \lim_{r_0 \to \infty} [r_0 p(r_0)],$$

so that

$$f_\infty(\bar{r}_0) = \frac{1}{4\pi} \exp(-jkr_0) \int_S [jk(\hat{n}' \cdot \hat{r}_0)\, p(\bar{r}_s) - \omega^2 \rho w(\bar{r}_s)] \exp(jk\hat{r}_0 \cdot \bar{r}_s)\, ds'. \quad (9)$$

Once the integro-differential equation is solved, the far zone field is easily evaluated by performing the integration indicated above. The acoustic scattering cross section, σ, for a unit amplitude incident wave, as defined by

$$\sigma = 4\pi\, |f_\infty|^2 \quad (10)$$

is also easily determined.

C. *The Harmonic Expansion Approach*

An alternate method is included here for use in the validation of the integro differential equation (IDE) method. The harmonic expansion method is ideally suited to the problem since the shell surfaces conform with complete coordinate surfaces in a separable geometry and hence insure separability of the wave equation.

The theory of three-dimensional elasticity rather than the thin shell theory is used in this approach to the scattering problem since the wave behavior within the shell is explicitly considered. Hence the presence of compressional and shear waves is allowed and accounted for in the elastic material. On the other hand, in the IDE approach we considered only the outer surface effects as defined by the thin shell theory.

The complete details of the mathematical analysis are given elsewhere (Goodman *et al* 1960). The important features are that the wave equation is separable, that there is symmetry in the φ coordinate, and that the solution for the displacement vector can be written as

$$\bar{u} = \nabla\Phi + \nabla \times \bar{\psi},$$

where Φ is a scalar potential and $\bar{\psi}$ a vector potential. The solutions for the potentials, due to separability, can be written as

$$\Phi = \sum_{l=0}^{\infty} P_l(\cos\theta)[A_l j_l(kr) + B_l n_l(kr)],$$

and

$$\psi_\varphi = \sum_{l=0}^{\infty} \frac{\partial}{\partial\theta} P_l(\cos\theta)[C_l j_l(kr) + D_l n_l(kr)],$$

$$\psi_r = \psi_\theta = 0,$$

where P_l (cos θ) is the Legendre polynomial of the first kind and $j_l(kr)$ and $n_l(kr)$ are spherical Bessel functions of the first and second kind, respectively. Equations of this type can be written for each region but the ψ potential exists only where shear waves can be supported, which in our problem is in the shell only.

Since each expansion contains two unknown coefficients for each index l, it is necessary to specify boundary conditions in order to allow for their determination. Two of these conditions are provided by the radiation condition at infinity and the requirement that the field be finite at the origin. In addition to these conditions, it is also necessary that the displacement and the stress tensor be continuous at the interfaces. These specified conditions are sufficient for the determination of the coefficients in the expansions.

The quantities of most interest, i.e., the far-field pressure amplitude and the scattering cross section, are easily determined. The equations are written here for convenience; the derivations are found in Goodman *et al* (1960).

The scattering cross section of an elastic sphere is given by

$$\sigma = \frac{4\pi}{k^2} \left| \sum_{l=0}^{\infty} (-i)^{l+1} A_l^I P_l(\cos\theta) \right|^2 \tag{11}$$

where k is the propagation constant in the exterior fluid and A_l^I is the expansion coefficient of the lth order spherical Hankel function.

The far-field pressure amplitude is related to the scattering cross section and is given by

$$|f_\infty| = \sqrt{\frac{\sigma}{\pi a^2}} = \frac{2}{ka} \left| \sum_{l=0}^{\infty} (-i)^{l+1} A_l^I P_l(\cos\theta) \right| \tag{12}$$

Extensive numerical computations for elastic spheres and spherical shells based upon the harmonic expansion discussed above have been presented by Hickling (1962, 1964), and Diercks and Hickling (1967) and Hickling and Means (1968).

III. Numerical Solution Technique

A. *The General Procedure*

The solution method for the coupled equations previously developed follows that of the method of moments. The linear integral Eq. (5) can be written in operator notation as

$$L_p[p(\theta')] + L_w[w(\theta')] - 1/2p(\theta) + p_{\mathrm{inc}}(\theta) = 0, \tag{13}$$

where L_p and L_w represent the linear operators which act on p and w, respectively. The unknown functions can be represented by an expansion in basis or trial functions as

$$\begin{aligned} p(\theta') &\approx \sum_{n=1}^{N} a_n p_n(\theta'), \\ w(\theta') &\approx \sum_{n=1}^{N} b_n w_n(\theta'), \end{aligned} \tag{14}$$

where the a_n and b_n are constants to be determined and the functions $p_n(\theta')$ and $w_n(\theta')$ are independent in the domain of the operator. A residual error can be defined [Fenlon (1969)] as

$$e(x) = \sum_{n=1}^{N} \{a_n(L_p[p_n(\theta')] - \tfrac{1}{2}p_n(\theta)) + b_n L_w[w_n(\theta')]\} + p_{\mathrm{inc}}(\theta), \tag{15}$$

where the linearity of L_p and L_w has been used to interchange the summation and integration. If one defines an inner product over a surface S of two functions X and Y as

$$\langle X, Y\rangle = \iint_S XY\,da$$

one can, by taking the inner product of Equation (15) with a set of M weighting or testing functions $\{t_m\}$ in the range of the operator L, write

$$\langle t_m, e\rangle \equiv \sum_{n=1}^{N} \{a_n\langle t_m, L_p[p_n(\theta')] - \tfrac{1}{2}p_n(\theta)\rangle + b_n\langle t_m, L_w[w_n(\theta')]\} + \langle t_m, p_{\mathrm{inc}}(\theta)\rangle$$
$$m = 1, 2, \ldots, M. \tag{16}$$

Furthermore, if the projection of the residual error on the space of the weight functions is set to zero, one has an equation which can be cast in the form

$$\mathsf{Z}^{(p)} \cdot \mathbf{A} + \mathsf{Z}^{(w)} \cdot \mathbf{B} = \mathbf{P}^{(\mathrm{inc})}, \tag{17}$$

where the elements of $\mathbf{Z}^{(p)}$ and $\mathbf{Z}^{(w)}$ are given by

$$Z_{mn}^{(p)} = \langle t_m , L_p[p_n(\theta')] - \tfrac{1}{2}p_n(\theta)\rangle$$
$$Z_{mn}^{(w)} = \langle t_m , L_w[w_n(\theta')]$$

and those of $\mathbf{P}^{(\text{inc})}$ are given by

$$P_m^{(\text{inc})} = -\langle t_m , p_{\text{inc}}(\theta)\rangle.$$

The original operator equation has thus been reduced to a linear system of M equations in $2N$ unknowns.

Equation (17) thus replaces the integral Eq. (5) for the purpose of our numerical solution. A similar treatment of Eq. (8) [or of Eqs. (6) and (7)] will then complete the reduction of the integro-differential equation to linear system form. Substitution of (14) into (8) leads to

$$\mathbf{Z}^{(a)} \cdot \mathbf{A} = -\mathbf{Z}^{(b)} \cdot \mathbf{B}, \tag{18}$$

where

$$Z_{mn}^{(a)} = \left\langle t_m , \left[E\left(\frac{d^2}{d\theta^2} + \cot\theta\right) + D\right] w_n\right\rangle,$$

$$Z_{mn}^{(b)} = \left\langle t_m , \left[F\left(\frac{d^2}{d\theta^2} + \cot\theta\right) + B\right] p_n\right\rangle.$$

Equations (17) and (18) together represent then $2M$ equations in $2N$ unknowns so that with $M = N$ a deterministic linear system is obtained for the $2N$ constants contained in **A** and **B**. This system can be simultaneously solved for **A** and **B**; for example, **A** could be found from Eq. (18) in terms of **B** after which **B** is obtained from Eq. (17).

Fenlon (1969) has tabulated some of the more common pairs of functions used for solving integral equations. In the present analysis a subsectional collocation technique was used. In this particular method, the weight functions were chosen to be a set of delta functions,

$$t_m = \delta(\theta - \theta_m)\ ;\ m = 1,\ldots, N$$

and the basis were chosen as

$$p_n(\theta) = \begin{cases} 1, & \theta \in \Delta\theta_n\ ; \\ 0 & \text{otherwise} \end{cases} \quad n = 1,\ldots, N, \tag{19}$$

where $\Delta\theta_n$ is a small interval centered at θ_n.

Note that the union of all the $\Delta\theta_n$ (which are all taken to be equal of value $\Delta\theta$) covers the domain of the operator. For instance, if the function $p(\theta)$ is a continuously varying function, then the series in Eq. (14) with the basis functions given by Eq. (19) approximates $p(\theta)$ by a series of steps.

The interaction matrix elements in Eq. (17), $Z^{(p)}$ and $Z^{(w)}$ are thus obtained by straightforward numerical integration or quadrature over azimuthal strips of width $\Delta\theta$. Similarly, the elements of $Z^{(a)}$ and $Z^{(b)}$ which appear in (18) are found using a finite difference form for the differential operators which appear in them. In essence then, the approach described here results in enforcing the integro-differential equation at N discrete points at which p and w are evaluated.

B. *Specifics of the Present Problem*

In order to implement the procedures described, we first subdivide the sphere into a number (N) of bands as shown in Fig. 3 such that the center of each band is given by

$$\theta_i = (i - 1/2)\,\pi/N, \qquad i = 1, 2,\ldots, N \tag{20}$$

and the width is given by $\Delta\theta = \pi/N$.

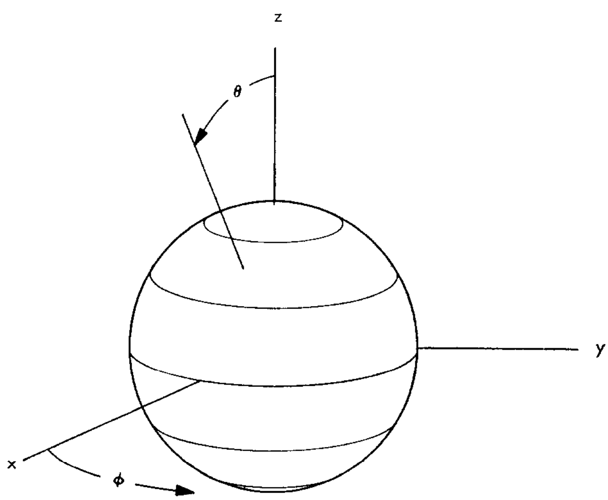

FIG. 3. Subdivision of the spherical scattering surface.

Since the pressure and velocity on the surface are independent of azimuthal coordinate (φ),

$$p(a, \theta', \varphi') \to p(a, \theta'),$$

$$w(a, \theta', \varphi') \to w(a, \theta'),$$

and since we are using the pulse approximation for the functions within each band, we can write

$$p(a, \theta') = \sum_{i=1}^{N} p_i U(\theta_i'),$$
$$w(a, \theta') = \sum_{i=1}^{N} w_i U(\theta_i'), \tag{21}$$

where

$$U(\theta_i') = 1, \qquad \theta_i - \frac{\Delta\theta}{2} < \theta' < \theta_i + \frac{\Delta\theta}{2},$$
$$= 0, \qquad \text{elsewhere.}$$

By choosing the weight functions to be as defined in Eq. (18), and by substituting Eqs. (5), (18) and (21) into Eq. (16) with $\langle t_m, e\rangle = 0$, one arrives at

$$-4\pi p_i^{\text{inc}} = -2p_i + \sum_{j=1}^{N} 2a^2 \sin\theta_j \sin\frac{\Delta\theta}{2}$$
$$\times \left\{ p_j \,⨍_0^{2\pi} \frac{\partial}{\partial n'} \frac{e^{-jkR_{ij}}}{R_{ij}}\, d\varphi' - \omega^2 \rho w_j \,⨍_0^{2\pi} \frac{e^{-jkR_{ij}}}{R_{ij}}\, d\varphi' \right\}$$
$$i = 1, 2,..., N, \tag{22}$$

where

$$R_{ij} = |\bar{r}_0(\theta_i) - \bar{r}_s(\theta_j, \varphi')|$$
$$= [2a^2(1 + \sin\theta_i \sin\theta_j \cos\varphi' + \cos\theta_i \cos\theta_j)]^{1/2}.$$

Equation (22) represents N simultaneous linear equations of the 2 N unknowns, p_i and w_i. The evaluation of the integrals over the azimuthal coordinate can be carried out using a standard quadrature technique. In the band containing the singularity, the principal value nature of the integral must be taken into account. This can be accomplished in practice by considering

$$⨍_S (\ )\, d\varphi' = \int_{S-\Delta S} (\ )\, d\varphi'$$

and by evaluating the integral in the limit $\Delta S \to 0$ to ensure that it approaches a constant limiting value.

The introduction of additional equations to provide a consistent system for solution can proceed in a similar fashion. Evaluation of Eq. (8) at the prescribed θ_i ($i = 1, 2,..., N$) gives rise to an additional N equations in w and p.

We then have

$$E \frac{d^2w}{d\theta^2}\bigg|_{\theta_i} + E \cot \theta_i \frac{dw}{d\theta}\bigg|_{\theta_i} + Dw(\theta_i)$$

$$+ F \frac{d^2p}{d\theta^2}\bigg|_{\theta_i} + F \cot \theta_i \frac{dp}{d\theta}\bigg|_{\theta_i} + Bp(\theta_i) = 0, \qquad i = 1,..., N. \tag{23}$$

The differential operators can be represented by finite differences of any desired order. For instance, if we write

$$\frac{d^2w}{d\theta^2}\bigg|_{\theta_i} = \frac{1}{(\Delta\theta)^2} [w_{i+1} - 2w_i + w_{i-1}],$$

and

$$\frac{dw}{d\theta}\bigg|_{\theta_i} = \frac{1}{2\Delta\theta} [w_{i+1} - w_{i-1}],$$

then (23) becomes

$$\frac{E}{\Delta\theta}\left(\frac{1}{\Delta\theta} + \frac{\cot \theta_i}{2}\right) w_{i+1} + \left(-\frac{2E}{\Delta\theta^2} + D\right) w_i + \left(\frac{1}{\Delta\theta} - \frac{\cot \theta_i}{2}\right) w_{i-1}$$

$$= \frac{F}{\Delta\theta}\left(\frac{1}{\Delta\theta} + \frac{\cot \theta_i}{2}\right) p_{i+1} + \left(-\frac{2F}{\Delta\theta^2} + B\right) p_i + \left(\frac{1}{\Delta\theta} - \frac{\cot \theta_i}{2}\right) p_{i-1},$$

$$i = 1, 2,..., N. \tag{24}$$

The coupled set of equations represented by Eqs. (22) and (23) are easily written in matrix form so that they can be solved numerically for the surface pressure and normal displacement. This equation, in the form of (17), can then be solved for the unknown vector [A] by inversion, factorization, or iteration. The matrix [Z] has a dimensionality of $2N$, but the portion of the matrix associated with the differential equations is sparse. In fact of the $(2N)^2$ elements in [Z], only $2N^2 + 6N$ elements are nonzero. When solving the matrix equation this characteristic of [Z] should be taken into account since it can lead to a substantial reduction of the matrix fill and solution time.

The radiation field at a point $(r_0, \theta_0, \varphi_0)$ can be evaluated by reducing the integral in Eq. (9) to a summation over bands in the form

$$P_{\text{far field}}(r_0, \theta_0, \varphi_0) = \frac{1}{4\pi} e^{-jkr_0} \sum_{i=1}^{N} [j^k p_i I_1 - \omega^2 \rho w_i I_2] \left[2a^2 \sin \theta_i \sin \frac{\Delta\theta}{2}\right], \tag{25}$$

where

$$I_1 = [j2\pi AJ_1(kaA) + 2\pi BJ_0(kaA)]\, e^{jkaB},$$
$$I_2 = 2\pi J_0(kaA)\, e^{jkaB},$$
$$A = \sin\theta_i \sin\theta_0\,,$$
$$B = \cos\theta_i \cos\theta_0\,,$$

with J_0 and J_1 representing Bessel functions of zeroeth and first order, respectively. Note that the above equation is independent of φ_0 since the problem possesses azimuthal symmetry. The scattering cross section is then the squared magnitude of the above multiplied by 4π. Similarly, the far field pressure amplitude $|f_\infty|$ is determined from $|f_\infty| = \sqrt{\sigma/\pi a^2}$.

IV. Numerical Results

As mentioned above, the analytical treatment has been included in order to provide independent data for validating the IDE calculations. Results obtained using the analytical approach (harmonic expansions) have been extensively tested against published data available in the literature. The agreement which was realized in these tests allows full confidence to be placed in the computer program for the harmonic expansion approach to the elastic body scattering problem.

The data to be presented is intended to illustrate the accuracy of the IDE approach. In all cases the matrix equation was solved using factorization, a procedure which results in a solution time proportional to $N^3/3$. The matrix elements corresponding to the integral equation were computed by using a Romberg variable interval width technique (Miller, 1970). In the bands containing the singularity this same technique was used but a region around the singularity was excluded. The contribution of this excluded region was then evaluated using a rectangular rule integration with the singular point removed. This scheme for handling the singularity was tested by changing the integration sample density around the point $\bar{r}_s = \bar{r}_0$ and by considering the stability of the result. It was found that for a square region around the singularity of width π/N, a sample density of nine (with the singular point removed) was sufficient.

Preliminary calculations using the IDE approach were performed for the rigid ($w = 0$) and free-surface ($p = 0$) sphere. In each case the dimensionality of the coupled Eqs. (22) and (24) becomes N, since these limiting cases correspond to an uncoupled system with either p or w as the only unknown in the integral equation. Furthermore, the differential equation becomes homogeneous in p or w under these conditions. The results obtained are shown in Fig. 4 compared with the published data of Goodman *et al.* (1966). The agreement is especially significant since it

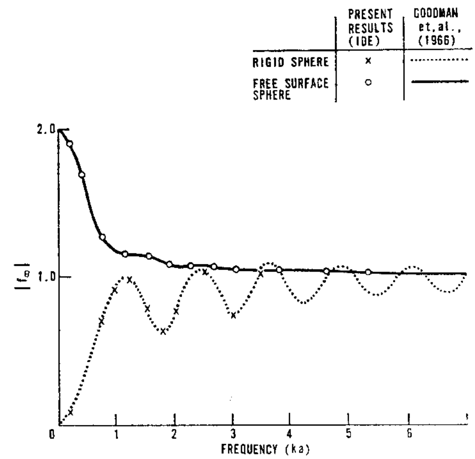

FIG. 4. Far-field pressure amplitude foı a rigid and free-surface sphere.

demonstrates the accuracy with which the integral equation, including the influence of the singularities, is solved. These calculations do not, however, establish the usefulness of the IDE approach for elastic shells, since the differential equations for the elastic shell are of course not involved in the rigid- and free-sphere cases. A series of additional calculations using the analytical and IDE approaches were carried out to further validate the IDE results for elastic shells.

Since the IDE approach is derived on the basis that $\Delta \ll a$, it is important in these initial calculations to establish the accuracy dependence of the IDE results on this ratio. We show in Figs. 5, 6 and 7 the acoustic cross section $\sigma/\pi a^2$, as a

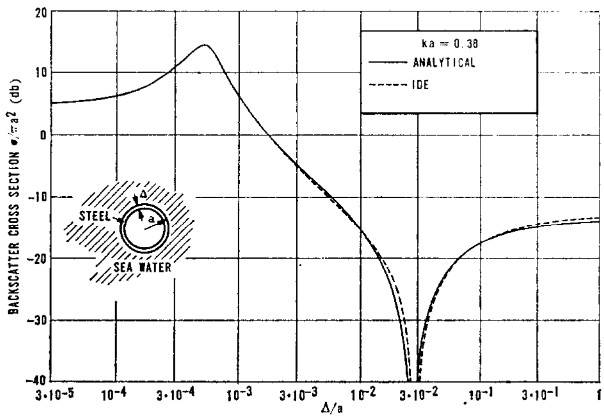

FIG. 5. Backscatter cross section versus normalized thickness of a fluid-immersed spherical shell for $ka = 0.38$.

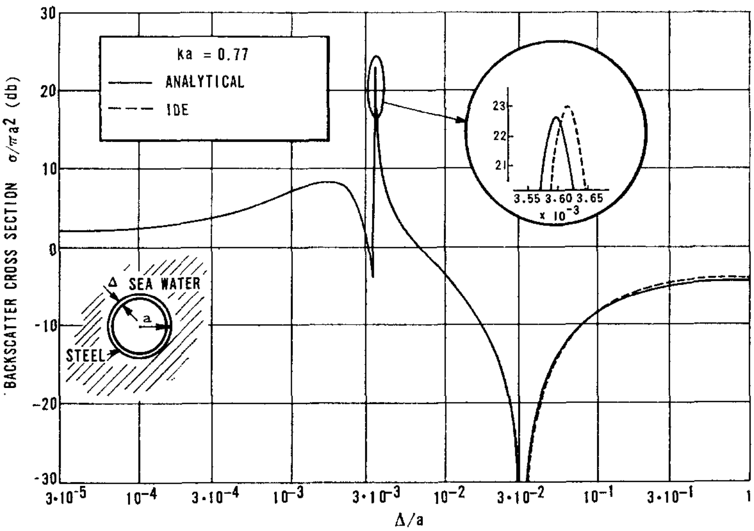

FIG. 6. Backscatter cross section versus normalized thickness of a fluid-immersed spherical shell for $ka = 0.77$.

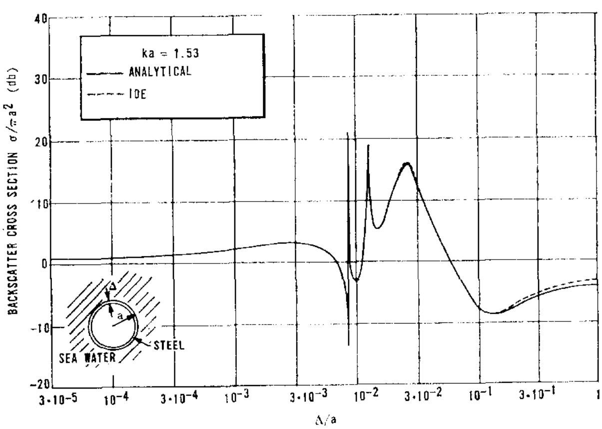

FIG. 7. Backscatter cross section versus normalized thickness of a fluid-immersed spherical shell for $ka = 1.53$.

function of Δ/a for steel shells with vacuum interiors and ka values of 0.38, 0.77 and 1.53, The shell parameters used were

$$Y = 19.67 \times 10^{11} \text{ dynes/cm}^2,$$
$$\nu = 0.3,$$
$$P_s = 7.70 \text{ g/cm}^3.$$

Those for sea water used were

$$\rho = 1.02 \, g/\text{cm}^3.$$

The data plotted here is presented in terms of the nondimensional quantities Δ/a, ka and $\sigma/\pi a^2$, since the scattering properties of bodies with frequency independent acoustic parameters are scalable.

The analytical values are shown by the solid curves while results obtained from the IDE approach are shown by dashed curves where they differ enough from the analytic data to be graphically resolved. It may be seen that the IDE results are in excellent agreement with the analytic values, both in the resonant peak structure as well as for values of Δ/a approaching 1, where it could be reasonably expected that

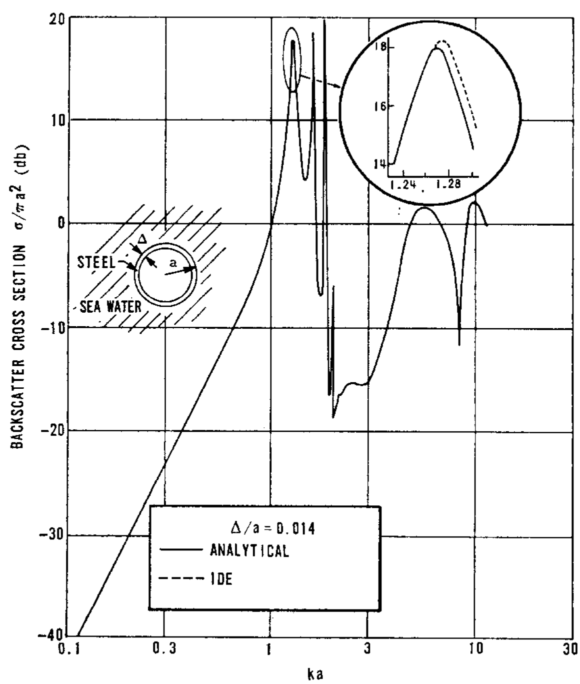

FIG. 8. Backscatter cross section versus ka of a fluid-immersed spherical shell of normalized thickness 0.014.

the thin shell approximation made in the IDE approach would be incapable of accurately predicting the cross section. Possibly the special symmetry of the shell is responsible for the success of the IDE method for $\Delta/a \sim 1$; we cannot safely extrapolate this finding to other shell shapes, however.

Elastic shell cross section results versus normalized frequency ka are shown in Fig. 8 where the independent variable is now the incident wave frequency expressed in the shell circumference in wavelengths ka, and Δ/a is the parameter. Again excellent agreement is obtained between the two methods used to calculate $\sigma/\pi a^2$. It is interesting to see that for frequencies below $ka \simeq 0.3$, the Rayleigh region scattering law is verified, i.e., $\sigma/\pi a^2 \alpha (ka)^4$.

The numerical difficulties which often occur at eigenfrequencies of the interior problem have not been encountered in this effort. One reason for this is that computations were not performed at the eigenfrequencies in the free and rigid sphere cases. In the elastic sphere calculations the coefficient matrix in the matrix equation was never observed to be approaching a singular condition, an indication that the matrix solution process did not involve operations on an ill-conditioned matrix. A study of this point would certainly be of great use in future investigations.

V. Conclusions

The integro-differential equation approach has been applied to the problem of determining the scattering characteristics of the fluid-immersed elastic sphere. The technique has been shown to provide accurate results over a large range of shell thickness even though the thin shell approximation has been used in defining the pressure displacement relationship on the surface. The success of the IDE method suggests the applicability of the approach to the treatment of more general elastic structures, rotationally symmetric shells for example. Extension of the equations of motion to include bending moments as well as flexural shell motion for more complicated shell shapes will further increase the applicability of the method. By combining the IDE treatment with conventional vibration analysis techniques, it appears feasible to obtain the acoustic cross section of fairly complicated shell geometries, including the effects of internal structural characteristics. Any extension to more complicated structures will necessarily include a study of the integral equation sample density requirements.

Acknowledgment

The authors wish to acknowledge the assistance of Mr. G. D. Pegan in preparation of the computer programs for this analysis.

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